Latin Square Design

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Introduction

- A Latin square is $n \times n$ table filled with n different symbols in such a way that each symbol occurs exactly once in each row and exactly once in each column.
- Example may include

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Orthogonal array Representation

- If each entry of $n \times n$ Latin square is written as a triple (r, c, s) where r is the row, c is the column, and s is the symbol, we obtain a set of n^2 triples called the orthogonal array representation of the square.
- The orthogonal array representation of the first Latin Square design is

$$\{(1,1,1),(1,2,2),(1,3,3)...,(3,3,2)\}$$

- Where for example the triple (2,3,1) means that in row 2, column 3 there is the symbol 1.

Cont'd

- The definition of a Latin square can be written in terms of orthogonal arrays as follows:
    i. There are n^2 triples of the form (r, c, c) where $1 \leq r, c, s \leq n$
    ii. All of the pairs (r, c) are different and all the pairs (c, s) are different
- Many operations on a Latin square produce another Latin Square.

Latin Square Design

- A Latin square design is a method of placing treatments so that they appear in a balanced fashion within a square block or field.
- Treatments appear once in each row and column. Replicates are also included in this design.
  - Treatments are assigned at random within rows and columns, with each treatment once per row and once per column.
  - There are equal numbers of rows, columns, and treatments.
  - Useful where the experimenter desires to control variation in two different directions

Advantages of Latin Square design

- They handle the case when we have several nuisance factors and we either cannot combine them into a single factor or we wish to keep them separate.
- They allow experiments with a relatively small number of runs.

Disadvantages of Latin Square design

- The number of levels of each blocking variable must equal the number of levels of the treatment factor.
- The Latin square model assumes that there are no interactions between the blocking variables or between the treatment variable and the blocking variable.

Procedure to create a Latin Square Design

- An appropriate randomization strategy is as follows:
  i. Write down any Latin square of the required size
  ii. Randomize the order of the rows.
  iii. Randomize the order of the columns.
  iv. Randomize the allocation of treatments to the letters of the square.

Statistical Model

- The statistical model for this design is

$$y_{ijk} = \mu + \alpha_i + \tau_j + \beta_k + \epsilon_{ijk}, i, j, k = 1, ..., p$$

- y_{ijk} is the observation in the i^{th} row and k^{th} column for the j^{th} treatment
- μ is the overall mean
- α_i is the i^{th} row effect, τ_j is the j^{th} treatment effect, β_k is the k^{th} column effect and ϵ_{ijk} is a random error.

Cont'd

- The assumptions on the effect terms are

$$\sum_{i=1}^{p} \alpha_i = 0, \sum_{j=1}^{p} \tau_j = 0, \sum_{k=1}^{p} \beta_k = 0$$

- We are interested in testing for zero treatment effect that is

$$H_0 : \tau_1 = ... = \tau_p = 0$$

vs

$$H_1 : \tau_j \neq 0$$

ANOVA Table

Source	SS	DF	MS	F
Treatments	$\sum_{j=1}^{p} \frac{y_{.j.}^2}{p} - \frac{y_{...}^2}{p^2}$	p-1	MST	$\frac{MST}{MSE}$
Rows	$\sum_{i=1}^{p} \frac{y_{i...}^2}{p} - \frac{y_{i...}^2}{p^2}$	p-1	MSR	
Columns	$\sum_{k=1}^{p} \frac{y_{..k}^2}{p} - \frac{y_{...}^2}{p^2}$	p-1	MSC	
Error	Substraction	(p-1)(p-2)		
Total	$\sum_i \sum_j \sum_k y_{ijk}^2 - \frac{y_{...}^2}{p^2}$	$p^2 - 1$		

- If the test statistic $F > F_{\alpha,p-1,(p-2)(p-1)}$ the null is rejected

Example 1

The effect of five different ingredients (A, B, C, D, E) on the reaction time of a chemical process is being studied. Each batch of new material is only large enough to permit five runs to be made. Furthermore, each run requires approximately hours, so only five runs can be made in one day. The experimenter decides to run the experiment as a Latin square so that day and batch effects may be systematically controlled. She obtains the data that follow. Analyze the data from this experiment (use $\alpha = 0.05$) and draw conclusions.

Data

	Day				
Batch	1	2	3	4	5
1	A = 8	B=7	D=1	C=7	E=3
2	C=11	E=2	A=7	D=3	B=8
3	B=8	A=9	C=10	E=1	D=5
4	D=6	C=8	E=6	B=6	A=10
5	E=4	D=2	B= 3	A=8	C=8

Solution

- $p = 5$

	Day					
Batch	1	2	3	4	5	$y_{i..}$
1	A $=$ 8	B=7	D=1	C=7	E=3	26
2	C=11	E=2	A=7	D=3	B=8	31
3	B=4	A=9	C=10	E=1	D=5	29
4	D=6	C=8	E=6	B=6	A=10	36
5	E=4	D=2	B= 3	A=8	C=8	25
$y_{..k}$	33	28	27	25	34	$y_{...} = 147$

Cont'd

- $$SST = \sum_i \sum_j \sum_k y_{ijk}^2 - \frac{y_{...}^2}{p^2} = 1071 - \frac{147^2}{25} = 206.64$$

- $$SS_{Rows} = \sum_{i=1}^{p} \frac{y_{i..}^2}{p} - \frac{y_{...}^2}{p^2} = 879.8 - 864.36 = 15.44$$

- $$SS_{columns} = \sum_{k=1}^{p} \frac{y_{..k}^2}{p} - \frac{y_{...}^2}{p^2} = 876.6 - 864.36 = 12.24$$

- $$SS_{Treatments} = \sum_{j=1}^{p} \frac{y_{.j.}^2}{p} - \frac{y_{...}^2}{p^2} = 994.8 - 864.36 = 130.44$$

ANOVA Table

Source	SS	DF	MS	F
Treatment	130.44	4	32.61	8.07
Rows	15.44	4	3.86	
Columns	12.24	4	3.06	
Error	48.52	12	4.04	
Total	206.64	24		

- $F = 8.07 > F_{0.05,4,12} = 3.26$ we reject H_0 and conclude that there is a significance difference between the treatments.

Example 2

An industrial engineer is investigating the effect of four assembly methods (A, B, C, D) on the assembly time for a color television component. Four operators are selected for the study. Furthermore, the engineer knows that each assembly method produces such fatigue that the time required for the last assembly may be greater than the time required for the first, regardless of the method. That is, a trend develops in the required assembly time. To account for this source of variability, the engineer uses the Latin square design shown below. Analyze the data from this experiment ($\alpha = 0.05$) and draw appropriate conclusions.

Data

	Operator			
Order	1	2	3	4
1	C=10	D=14	A=7	B=8
2	B=7	C=18	D=11	A=8
3	A=5	B=10	C=11	D=9
4	D=10	A=10	B=12	C=14

Youden Squares

- Younden squares are incomplete Latin square designs in which the number of columns does not equal the number of rows and treatments.
- Youden square is a symmetric balanced incomplete block design in which rows correspond to blocks and each treatment occurs exactly once in each column or "position" of the block.

Example

- An example include

	Column			
Row	1	2	3	4
1	A	B	C	D
2	B	C	D	E
3	C	D	E	A
4	D	E	A	B
5	E	A	B	C

Statistical Model

- The linear model for a Youden square is

$$Y_{ijk} = \mu + \alpha_i + \tau_j + \beta_k + \epsilon_{ijk}$$

- The analysis of the Youden square is similar to the analysis of a balanced incomplete block design, except that a sum of squares between the position totals may also be calculated.

Example

In one of the experiments, the experimenter is interested in making comparisons among 7 treatments and there are 28 experimental units available. These 28 experimental units are arranged in a Youden Square design with 4 rows and 7 columns with one observation per row.

2(4.00)	3(5.30)	4(1.0)	5(16.9)	6(16.9)	7(10.3)	1(294)
7(17.5)	1(222)	2(12.20)	3(15.5)	4(11)	5(26.5)	6(27.2)
6(37)	7(26)	1(310)	2(22.7)	3(24.2)	4(21.4)	5(31.2)
5(46.8)	6(44.2)	7(34.3)	1(282)	2(33.7)	3(33.7)	4(30.50)

Solution

Rows	3	3319.006	1106.33	4.89
Columns	6	27595.41	4599.23	20.31
Treatment	6	195027.72	32504.62	143.56
Error	12	2717.08	226.42	
Total	27	228659.22		

Exercise

- Discuss the following types of lattice designs
  - Rectangular Lattice Designs
  - Cubic Lattice Designs

Thank You!